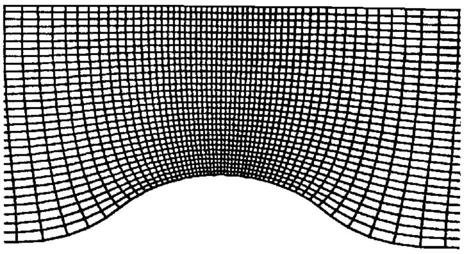
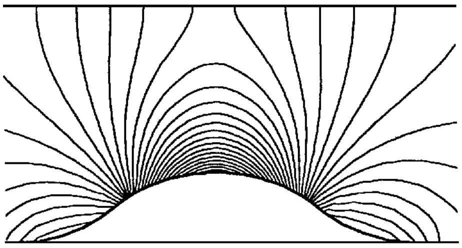


a) Streamline grid



b) Mach number contours

Fig. 2 Subsonic flowfield in the axisymmetric bumpy channel.

was calculated using central difference approximation and point Gauss–Seidel iterative algorithm with multigrid acceleration.

The implementation details of the numerical algorithms used to compute the flows presented in Figs. 1 and 2, as well as a generalization of the coordinate transformation Eqs. (2) and (3) and applications to steady supersonic vortical flow calculations, can be found in Ref. 9.

IV. Conclusions

The new formulations for compressible potential flow equations presented here utilize an orthogonal streamline-aligned system of independent natural, body-fitting coordinates. An advantage of the presented formulations over the similar previously known ones[1,3–5] is that the formulations developed in this work are expressed as meaningful differential conservation laws, which allows use of them to obtain conservative finite volume approximations.

References

[1]Barron, R. M., An, C.-F., and Zhang, S., "Survey of the Streamfunction-as-a-Coordinate Method in CFD," *Proceedings of the Inaugural Conference of the CFD Society of Canada*, CFD Society of Canada, Montreal, PQ, Canada, 1993, pp. 325–336.

[2]Liou, M.-S., "An Extended Lagrangian Method," *Journal of Computational Physics*, Vol. 118, No. 2, 1995, pp. 294–309.

[3]Dulikravich, G. S., "A Stream-Function-Coordinate (SFC) Concept in Aerodynamic Shape Design," AGARD-R-780, Brussels, Belgium, 1990, pp. 6.1–6.6.

[4]Stanitz, J. D., "Design of Two-Dimensional Channels with Prescribed Velocity Distributions Along the Channel Walls," NACA TN 2595, May 1952.

[5]Osipov, I. L., Shipilin, A. V., and Shulishnina, N. P., "A Numerical Method for Calculation of Gas Flows in Channels and Nozzles in Direct, Inverse and Combined Modes," *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki*, Vol. 27, No. 10, 1987, pp. 1563–1572 (in Russian).

[6]Latypov, A. M., and Shipilin, A. V., "On an Iterational Method for Calculating Sub- and Transonic Internal Flows," *USSR Computational Mathematics and Mathematical Physics*, Vol. 31, No. 5, 1991, pp. 92–99.

[7]von Mises, R., "Bemerkungen zur Hydrodynamik," *ZAMM*, Vol. 7, 1927, p. 425.

[8]Ovsyannikov, L. V., *Lectures on Fundamentals of Gas Dynamics*, Nauka, Moscow, 1981, Chap. 4 (in Russian).

[9]Latypov, A. M., *Numerical Solution of Steady Euler Equations in Streamline-Aligned Orthogonal Coordinates*, IMA Preprint Series No. 1182, Inst. for Mathematics and Its Applications, Univ. of Minnesota, Minneapolis, MN, Nov. 1993, pp. 1–33.

D. S. McRae
Associate Editor

Vorticity Jump in Surface Coordinates Across a Shock in Nonsteady Flow

Z. U. A. Warsi*
Mississippi State University,
Mississippi State, Mississippi 39762

Introduction

RECENT advances in various areas of computational fluid dynamics (CFD) have prompted the use of the formulas developed by Hayes[1] for the calculation of vorticity jump across a shock. Very recently an interesting discussion of Hayes' work by Isom and Kalkhoran[2] and Emanuel[3] has appeared, which further tends to establish the correctness of the work reported in Ref. 1. The simplicity and elegance of the formulas derived by Hayes are due to his choice of the normal coordinate as straight lines in both the steady and nonsteady flow situations. Obviously curved normal coordinates can be chosen equally but their introduction will involve unnecessary algebra with no change in the jump condition results.

The purpose of this Note is to obtain Hayes' formulas in the coordinates attached to an arbitrarily moving and deforming shock surface that occurs in nonsteady flows. The formulas given here can be incorporated in a flow solver that uses a coordinate system attached to a moving and deforming shock surface. Further, the essential geometrical properties of the shock surface are shown to depend on quantities that are directly computable.

In the present problem, the geometry of the shock surface changes with time and, therefore, a coordinate generator has to be used for the generation of surface coordinates and also for the calculation of the elements of the surface curvature tensor. It seems that the surface coordinate generator as developed by Warsi[4] (refer also to Ref. 5, p. 649) is most suitable for the present purpose inasmuch as the coordinate generation equations in Ref. 4 explicitly depend on the geometrical properties of the surface, and they also satisfy other differential-geometric properties.[6]

Analysis

Let x^i, $i = 1, 2, 3$, be a general time-dependent coordinate system such that x^α, $(\alpha = 1, 2)$, is a general coordinate system in the shock surface and $x^3 = n$ the actual normal distance from the shock surface at any given time. Thus, the coordinates form a local parallel surface coordinate system at a given instant. Let $\boldsymbol{r} = (x, y, z)$ be a point on the shock surface, then $\boldsymbol{a}_\alpha = \partial \boldsymbol{r}/\partial x^\alpha$, $\alpha = (1, 2)$, are the covariant base vectors in the shock surface, and the grad or nebla operator (repeated indices implying a sum) is

$$\text{grad} = \boldsymbol{a}^\alpha \frac{\partial}{\partial x^\alpha} + \boldsymbol{n}\frac{\partial}{\partial n} \tag{1a}$$

where $\boldsymbol{n}$ is the unit normal vector at the surface drawn along the direction of increasing n. Thus,

$$\begin{aligned} \boldsymbol{a}^\alpha &= \text{grad}\, x^\alpha, & \boldsymbol{a}^3 &= \boldsymbol{a}_3 = \boldsymbol{n} \\ g^{13} &= g_{23} = 0, & g_{33} &= g^{33} = 1 \end{aligned} \tag{1b}$$

(Ref. 5, p. 579), where g_{ij} and g^{ij} are the covariant and the contravariant metric coefficients, respectively. In the ensuing analysis, we have adopted the nondyadic operation for the gradient of vectors and divergence of tensors (Ref. 5, p. 602), i.e.,

$$\text{grad}\, \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x^\alpha}\boldsymbol{a}^\alpha + \frac{\partial \boldsymbol{u}}{\partial n}\boldsymbol{n} \tag{1c}$$

Received Feb. 3, 1997; revision received March 31, 1997; accepted for publication April 3, 1997.

*Professor, Department of Aerospace Engineering. Member AIAA.

whereas the curl of a vector is defined in the usual way:

$$\operatorname{curl} \boldsymbol{u} = \boldsymbol{a}^{\alpha} \times \frac{\partial \boldsymbol{u}}{\partial x^{\alpha}} + \boldsymbol{n} \times \frac{\partial \boldsymbol{u}}{\partial n} \tag{1d}$$

Because at any time t the shock surface is defined as $n =$ const, the unit normal $\boldsymbol{n}$ is

$$\boldsymbol{n} = \operatorname{grad} n \tag{2}$$

with

$$|\operatorname{grad} n|^2 = g^{33} = 1$$

From Eq. (2) we readily conclude that

$$\operatorname{curl} \boldsymbol{n} = 0 \tag{3}$$

so that

$$\boldsymbol{a}^{\alpha} \times \frac{\partial \boldsymbol{n}}{\partial x^{\alpha}} + \boldsymbol{n} \times \frac{\partial \boldsymbol{n}}{\partial n} = 0$$

Using the formulas of Weingarten (e.g., Ref. 5, p. 645),

$$\frac{\partial \boldsymbol{n}}{\partial x^{\alpha}} = -b_{\alpha\beta} \boldsymbol{a}^{\beta}$$

where $b_{\alpha\beta} = b_{\beta\alpha}$ are the coefficients of the second fundamental form, we have

$$\boldsymbol{a}^{\alpha} \times \frac{\partial \boldsymbol{n}}{\partial x^{\alpha}} = 0$$

Thus,

$$\boldsymbol{n} \times \frac{\partial \boldsymbol{n}}{\partial n} = 0$$

and $\boldsymbol{n}$ being a unit vector,

$$\boldsymbol{n} \cdot \frac{\partial \boldsymbol{n}}{\partial n} = 0$$

so that

$$\frac{\partial \boldsymbol{n}}{\partial n} = 0 \tag{4}$$

We consider x^{α}, n as time-dependent coordinates, i.e.,

$$x^{\alpha} = x^{\alpha}(\boldsymbol{r}, t), \qquad n = n(\boldsymbol{r}, t), \qquad \tau = t$$

and

$$\boldsymbol{r} = \boldsymbol{r}(x^{\alpha}, n, \tau), \qquad t = \tau$$

Writing $\boldsymbol{r}_{\tau} = \partial \boldsymbol{r} / \partial \tau$, we write the following formulas from Ref. 5 (pp. 92–94):

$$\boldsymbol{r}_{\tau} + \boldsymbol{w} = 0 \tag{5a}$$

$$\frac{\partial}{\partial t}() = \frac{\partial}{\partial \tau}() + [\operatorname{grad}()] \cdot \boldsymbol{w} \tag{5b}$$

$$\frac{\partial x^i}{\partial t} = -\boldsymbol{r}_{\tau} \cdot \boldsymbol{a}^i \tag{5c}$$

From Eq. (5c)

$$\frac{\partial n}{\partial t} = -\boldsymbol{r}_{\tau} \cdot \boldsymbol{n} = -q_s \tag{5d}$$

where q_s is the velocity of the surface in the normal direction. Using Eq. (1c), a second-order tensor is defined as

$$\boldsymbol{R} = -\operatorname{grad} \boldsymbol{n} \tag{6}$$

which is the surface curvature tensor. Using the identity $\operatorname{grad}(\boldsymbol{n} \cdot \boldsymbol{n}) = 0$ and Eq. (3), we have

$$(\operatorname{grad} \boldsymbol{n}) \cdot \boldsymbol{n} = 0$$

so that $\boldsymbol{R}$ is a surface tensor. Thus,

$$\boldsymbol{R} = -\operatorname{grad} \boldsymbol{n} = -\frac{\partial \boldsymbol{n}}{\partial n^{\alpha}} \boldsymbol{a}^{\alpha}$$

which, by using the Weingarten formulas, is written in covariant components as

$$\boldsymbol{R} = b_{\alpha\beta} \boldsymbol{a}^{\beta} \boldsymbol{a}^{\alpha} \tag{7a}$$

in mixed components as

$$\boldsymbol{R} = b_{\alpha\beta} g^{\beta\gamma} \boldsymbol{a}_{\gamma} \boldsymbol{a}^{\alpha} \tag{7b}$$

and in contravariant components as

$$\boldsymbol{R} = b_{\alpha\beta} g^{\beta\gamma} g^{\alpha\delta} \boldsymbol{a}_{\gamma} \boldsymbol{a}_{\delta} \tag{7c}$$

The formulas (7a) and (7c) clearly show that $\boldsymbol{R}$ is a symmetric tensor.

Let $\boldsymbol{q}$ be the absolute fluid velocity. Following Hayes,[1] we write

$$\boldsymbol{q} = \boldsymbol{q}_t + \boldsymbol{n} q_n \tag{8a}$$

and the vorticity $\boldsymbol{\zeta}$ as

$$\boldsymbol{\zeta} = \operatorname{curl} \boldsymbol{q} = \boldsymbol{\zeta}_t + \boldsymbol{n} \zeta_n \tag{8b}$$

where the subscript t denotes the quantity tangential to the surface. The velocity relative to the moving coordinates is denoted as $\boldsymbol{v}$, which is

$$\boldsymbol{v} = \boldsymbol{q} + \boldsymbol{w} = \boldsymbol{q} - \boldsymbol{r}_{\tau}$$

which is also written as

$$\boldsymbol{v} = \boldsymbol{v}_t + \boldsymbol{n} q_r, \qquad \boldsymbol{v}_t = \boldsymbol{q}_t + \boldsymbol{w}_t$$

where

$$q_r = q_n - q_s$$

Using some basic vector identities (e.g., Ref. 5, p. 585), it is a straightforward matter to establish the following:

$$q_n = \boldsymbol{q} \cdot \boldsymbol{n} \tag{9a}$$

$$\boldsymbol{q}_t = \boldsymbol{n} \times (\boldsymbol{q} \times \boldsymbol{n}) \tag{9b}$$

$$\zeta_n = (\operatorname{curl} \boldsymbol{q}_t) \cdot \boldsymbol{n} \tag{9c}$$

and

$$\boldsymbol{\zeta}_t = \boldsymbol{n} \times \left(\frac{\partial \boldsymbol{q}_t}{\partial n} - \boldsymbol{R} \cdot \boldsymbol{q}_t - \frac{\partial q_n}{\partial x^{\alpha}} \boldsymbol{a}^{\alpha} \right) \tag{9d}$$

Also from Eq. (3.137) of Ref. 5,

$$-\operatorname{grad} p = \rho \frac{\partial \boldsymbol{q}}{\partial \tau} + \rho (\operatorname{grad} \boldsymbol{q}) \cdot \boldsymbol{v} \tag{10a}$$

where p is the pressure and ρ is the density. Using Eq. (2) of Ref. 1 and Eq. (5b), we get

$$\frac{\partial \boldsymbol{n}}{\partial \tau} = -\frac{\partial q_s}{\partial x^{\alpha}} \boldsymbol{a}^{\alpha} + \boldsymbol{R} \cdot \boldsymbol{w}_t \tag{10b}$$

Using Eq. (10b) in Eq. (10a), the surface gradient of p is obtained as

$$\begin{aligned} -\frac{\partial p}{\partial x^{\alpha}} \boldsymbol{a}^{\alpha} &= \rho (q_r + q_s) \left(-\frac{\partial q_s}{\partial x^{\alpha}} \boldsymbol{a}^{\alpha} - \boldsymbol{R} \cdot \boldsymbol{q}_t \right) \\ &\quad + \rho q_r \frac{\partial \boldsymbol{q}_t}{\partial n} + \rho \left(\frac{\partial \boldsymbol{q}_t}{\partial \tau} + \frac{\partial \boldsymbol{q}_t}{\partial x^{\alpha}} \boldsymbol{a}^{\alpha} \cdot \boldsymbol{v}_t \right) \end{aligned} \tag{11}$$

where, as already noted, $q_r = q_n - q_s$ is the relative normal velocity.

Using the jump equations [Eqs. (24) and (25) of Ref. 1],

$$\delta(\rho q_r) = 0 \tag{12a}$$

(where $\rho q_r = \rho_2 q_{r2} = \rho_1 q_{r1}$),

$$-\delta p = \rho q_r \delta q_n \tag{12b}$$

and

$$\delta\left(\frac{\partial p}{\partial x^{\alpha}} \boldsymbol{a}^{\alpha} \right) = \frac{\partial}{\partial x^{\alpha}} (\delta p) \boldsymbol{a}^{\alpha}$$

with Eq. (11), we get the Hayes' formulas for the vorticity jump, which are

$$\delta(\zeta_n) = 0 \tag{12c}$$

$$\delta\boldsymbol{\zeta}_t = \boldsymbol{n} \times \left\{ \delta\left(\frac{1}{\rho}\right) \frac{\partial}{\partial x^\alpha}(\rho q_r)\boldsymbol{a}^\alpha - \frac{\delta(\rho)}{\rho q_r} \times \left[\left[\frac{\partial \boldsymbol{q}_t}{\partial \tau} + \frac{\partial \boldsymbol{q}_t}{\partial x^\alpha}\boldsymbol{a}^\alpha \cdot \boldsymbol{v}_t - q_s \frac{\partial q_s}{\partial x^\alpha}\boldsymbol{a}^\alpha - (\boldsymbol{R} \cdot \boldsymbol{q}_t) q_s \right] \right] \right\} \tag{12d}$$

As should be expected, Eq. (12d) can be reduced to Eq. (30) of Ref. 1 when $\partial/\partial\tau$ is expressed in terms of $\partial/\partial t$ through Eq. (5b). Note that only the surface tangential part of the derivatives of $\boldsymbol{q}_t$ have to be retained in Eq. (12d). As has been noted by Hayes[1] and also by Isom and Kalkhoran,[2] the importance of Eq. (12d) lies in the fact that it is independent of any thermodynamic law.

As has been noted earlier, the surface curvature tensor $\boldsymbol{R}$ given in Eqs. (7) is symmetric. If, however, one uses the representation (7b), then the matrix formed by the mixed components is a nonsymmetric matrix, i.e.,

$$\boldsymbol{R} = \begin{bmatrix} b_{1\beta}g^{\beta 1} & b_{2\beta}g^{\beta 1} \\ b_{1\beta}g^{\beta 2} & b_{2\beta}g^{\beta 2} \end{bmatrix}$$

The use of the preceding matrix directly yields the sum of the principal curvature and the Gaussian curvature of the shock surface, i.e.,

$$\begin{aligned} \operatorname{tr}(\boldsymbol{R}) &= k_I + k_{II}, && \text{sum of the principal curvatures} \\ \det(\boldsymbol{R}) &= K, && \text{Gaussian curvature} \end{aligned} \tag{13}$$

From Warsi,[7] the physical components of $\boldsymbol{R}$ are

$$R(\beta\delta) = (g_{\beta\beta}/g_{\delta\delta})^{\frac{1}{2}} R_{\alpha\delta}g^{\alpha\beta} \tag{14}$$

(sum on α, and $R_{\alpha\delta} = b_{\alpha\delta}$). The matrix formed by the components given in Eq. (14) is again a nonsymmetric matrix, and its trace and determinant are the same as stated in Eq. (13). The terms appearing in Eqs. (13) and (14) can directly be computed by the coordinate generation program.

Conclusion

Hayes' vorticity jump conditions have been obtained in general shock surface oriented coordinates. The main formula (12d) exhibits Hayes' tangential vorticity jump formula in time and space coordinates attached to the shock surface. It can be incorporated in a flow code that uses moving and deforming coordinates attached to a shock. Because the coordinates are time dependent, they have to be regenerated at every time step. Note that $\boldsymbol{R}$ is a function of time because the coefficients of the first and the second fundamental forms ($g_{\alpha\beta}$, $b_{\alpha\beta}$) are functions of time.

References

[1] Hayes, W. D., "The Vorticity Jump Across a Gasdynamic Discontinuity," *Journal of Fluid Mechanics*, Vol. 2, Aug. 1957, pp. 595–600.

[2] Isom, M. P., and Kalkhoran, I. M., "Comment on the Vorticity Jump Across a Shock Wave," *AIAA Journal,* Vol. 34, No. 12, 1996, p. 2640.

[3] Emanuel, G., "Reply to M. P. Isom and I. M. Kalkhoran," *AIAA Journal,* Vol. 34, No. 12, 1996, p. 2641.

[4] Warsi, Z. U. A., "Numerical Grid Generation in Arbitrary Surfaces Through a Second-Order Differential Geometric Model," *Journal of Computational Physics,* Vol. 64, No. 1, 1986, pp. 82–96.

[5] Warsi, Z. U. A., *Fluid Dynamics: Theoretical and Computational Approaches*, CRC Press, Boca Raton, FL, 1993.

[6] Warsi, Z. U. A., "Theoretical Foundation of the Equations for the Generation of Surface Coordinates," *AIAA Journal,* Vol. 28, No. 6, 1990, pp. 1140–1142.

[7] Warsi, Z. U. A., "Operations on the Physical Components of Tensors," ZAMM, Vol. 76, No. 6, 1996, pp. 361–363.

W. Oberkampf
Associate Editor

Generalized Vortex Lattice Method for Planar Supersonic Flow

Paulo A. O. Soviero*
Instituto Tecnológico de Aeronáutica, São José dos Campos, Brazil
and
Hugo B. Resende†
Empresa Brasileira de Aeronáutica S.A., São José dos Campos, Brazil

Introduction

THE problem of a lifting surface oscillating harmonically in supersonic flow has been studied by several researchers of unsteady aerodynamics. However, if one considers classical linearized models, there is a wide number of available numerical methods of solution, implying that there is not a preferred, well-accepted way to solve for the pressure distribution over thin wings for a known movement. The reader is referred to Yates[1] for a complete review and discussion of the state of the art of unsteady aerodynamics. According to that work, the analysis problem of supersonic lifting surfaces may be classified into four numerical models. In the first one, called the integrated-pressure method, the acceleration potential is used as the main variable, and it is related to the downwash over the wing by means of an integral equation.[2–5] This model comes directly from the subsonic case and has similar methods of solution. The second model is based on the superposition of velocity potential sources over the wing, with the magnitudes given by an integral of the downwash, usually obtained numerically.[6,7] In this formulation, the upper and lower surfaces of the wing must be isolated from each other; that is, from a physical point of view, all the edges of the wing must be supersonic. If any edge is subsonic, there is a region where the lower and upper surfaces interact, and diaphragms must be employed. In the third and fourth models, called integrated-potential[8] and potential-gradient[9,10] methods, respectively, the velocity potential is used as the main variable and the integral equation relates the downwash to the velocity potential difference between the upper and lower surfaces of the wing and wake. Because the kernel function is related to a velocity potential source, and double differentiation normal to the planform is performed after integration, the main difference between those numerical methods comes from the way the kernel function is manipulated prior to integration. Finally, although the velocity potential formulations just reviewed are widespread, they are not immediately evident to the aerodynamicist because their well-known singular solutions as sources, doublets, and line vortices are hardly identifiable after so many integrations by parts, new variable definitions, or series expansions.

The present work considers the velocity potential formulation of supersonic, harmonically oscillating lifting surfaces. The problem is linearized and, therefore, limited to small disturbances and flat wakes, with the solution being obtained numerically. Both the wing and the wake are discretized by means of constant density doublet panels, and the complex potential results are obtained through the solution of the hyperbolic Helmholtz equation.[11] The equivalence between closed vortex loops and surfaces of constant density potential remains valid in the steady-state case,[12] leading to the not-so-widespread supersonic version[13] of the well-known incompressible vortex lattice method. Therefore, in the present work the generalized vortex lattice method for oscillating lifting surfaces in

Received April 22, 1996; presented as Paper 96-2491 at the AIAA 14th Applied Aerodynamics Conference, New Orleans, LA, June 17–20, 1996; revision received March 7, 1997; accepted for publication March 11, 1997.

*Professor, Aeronautical Engineering Division. Member AIAA.
†Aeroelastic Engineer, Structure Department. Member AIAA.